Example: Sketch the graph of the function $f(x) = x^2 e^{-x/5}$.

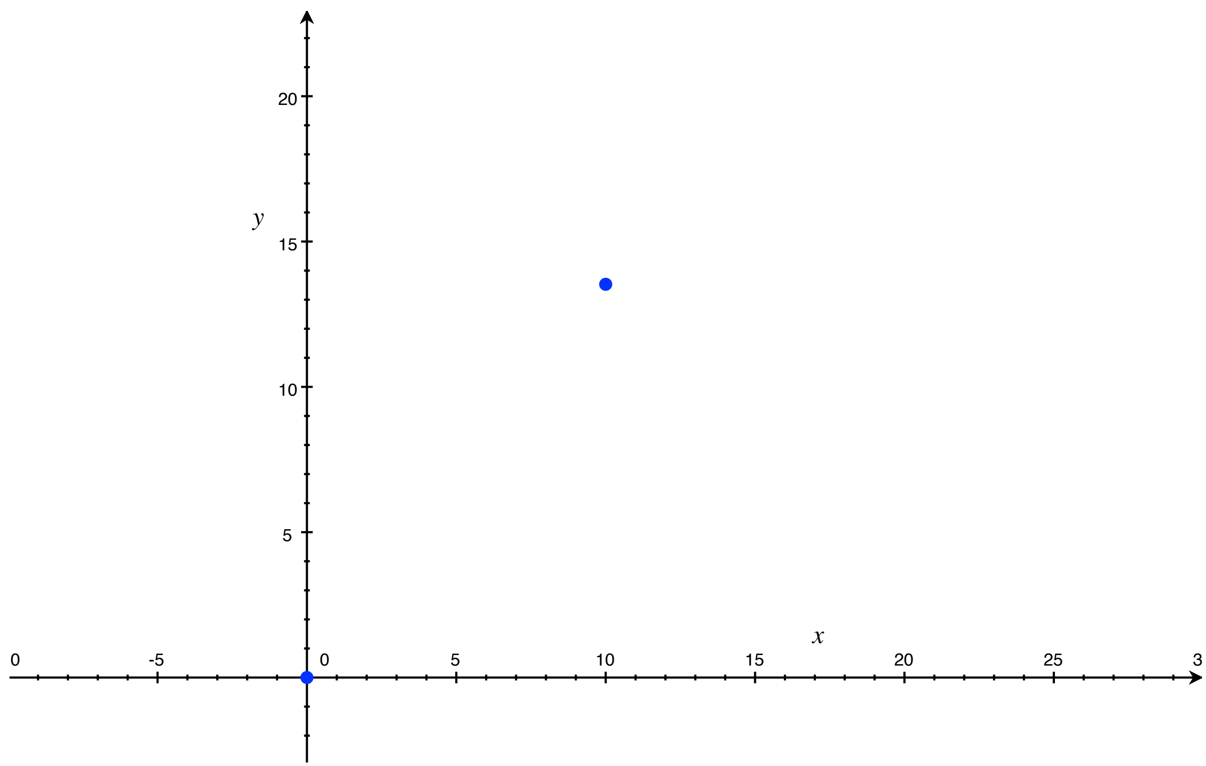
Step 1. Intercepts and critical points.

(*) $f(x) = 0 \implies x = 0 \ldots$ x-intercept and y-intercept: $(0, 0)$.

(*) $f'(x) = 0 \implies 2xe^{-x/5} - \frac{1}{5}x^2 e^{-x/5} = xe^{-x/5}(2 - \frac{1}{5}x) = 0$

$\implies x = 0$ or $x = 10$.

(*) Critical points: $(0, 0)$ and $(10, f(10)) \approx (10, 13.534)$.



Step 2. Intervals where the graph is increasing and decreasing.

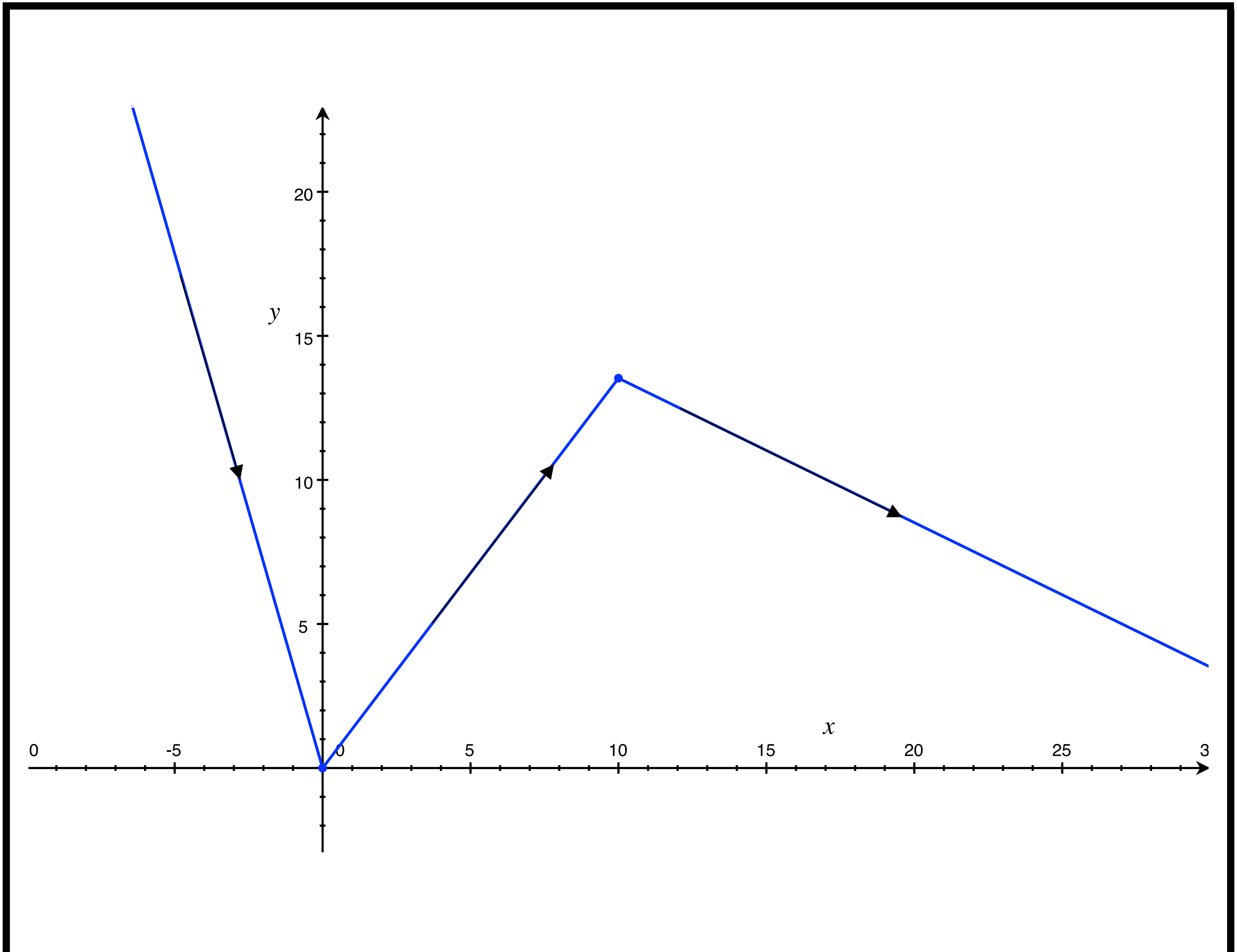
(*) Test the derivative to the left and right of critical points:

$$f'(x) = xe^{-x/5}(2 - \frac{1}{5}x)$$

(*) $f'(-1) = \overbrace{(-1)}^{-}\underbrace{e^{1/5}}_{+}\overbrace{(2 + \frac{1}{5})}^{+} < 0 \ldots$ so $f(x)$ is <u>decreasing</u> in $(-\infty, 0)$.

(*) $f'(1) = \overbrace{(1)}^{+}\underbrace{e^{-1/5}}_{+}\overbrace{(2 - \frac{1}{5})}^{+} > 0 \ldots$ so $f(x)$ is <u>increasing</u> in $(0, 10)$.

(*) $f'(11) = \overbrace{(11)}^{+}\underbrace{e^{-11/5}}_{+}\overbrace{(2 - \frac{11}{5})}^{-} < 0 \ldots$ so $f(x)$ is <u>decreasing</u> in $(10, \infty)$.



y
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x
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-5
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25
3

Step 3. Points of inflection and intervals where the graph is concave up and concave down.

(*) $f''(x) = \dfrac{d}{dx}\left(2xe^{-x/5} - \tfrac{1}{5}x^2e^{-x/5}\right)$

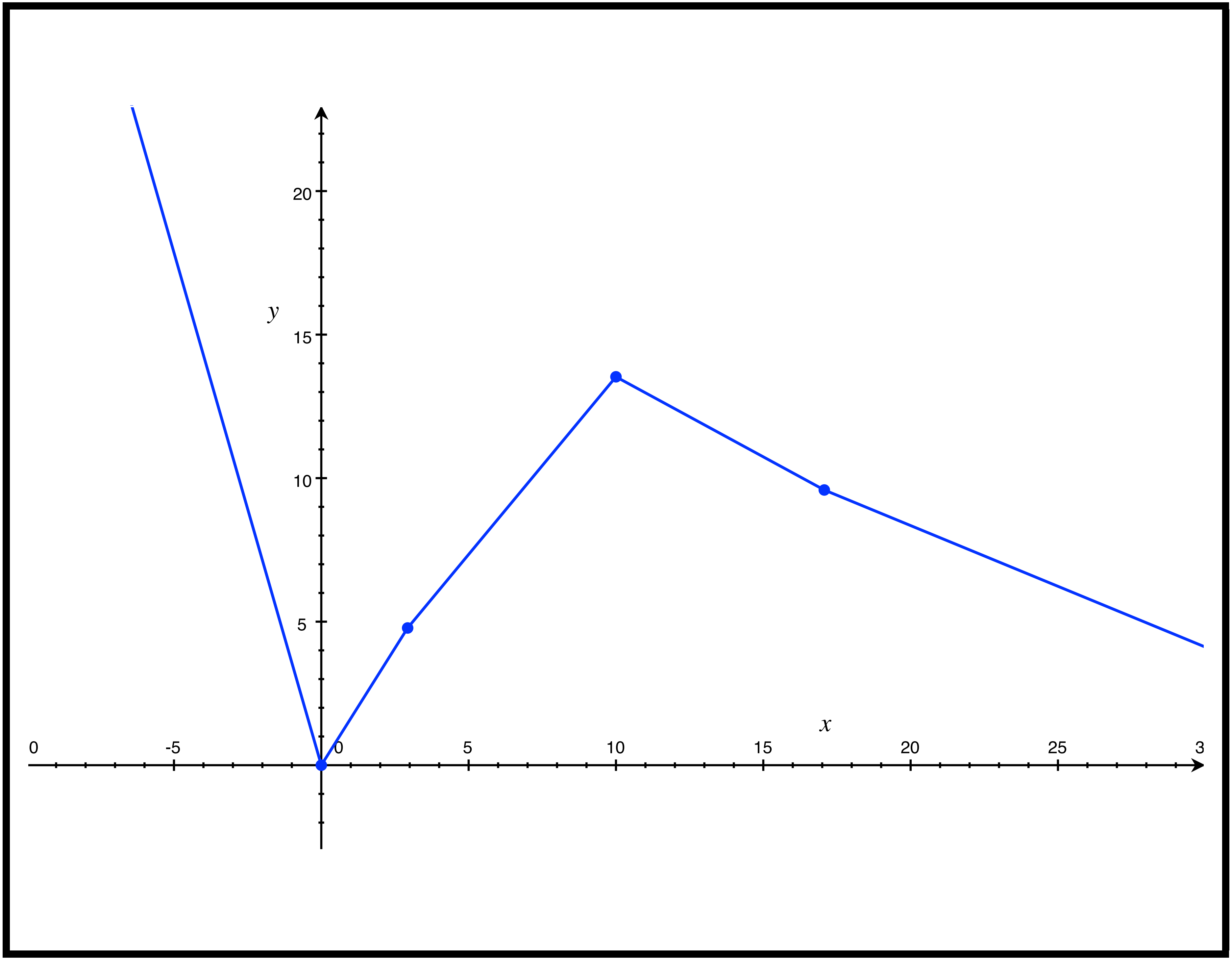
$$= 2e^{-x/5} - \tfrac{2}{5}xe^{-x/5} - \tfrac{2}{5}xe^{-x/5} + \tfrac{1}{25}x^2e^{-x/5}$$

$$= \tfrac{1}{25}e^{-x/5}\left(x^2 - 20x + 50\right)$$

(*) $f''(x) = 0 \implies x = \dfrac{20 \pm \sqrt{400-200}}{2} = 10 \pm \dfrac{\sqrt{200}}{2}$

$$\implies x = 10 - 5\sqrt{2} \approx 2.93 \quad \text{or} \quad x = 10 + 5\sqrt{2} \approx 17.07$$

(*) (Possible) Points of inflection: $(2.93, 4.78)$ and $(17.07, 9.59)$.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(*) Test the sign of $f''(x)$ to the left and right of the (possible) inflection points, to determine concavity.

(*) $f''(x) = \frac{1}{25}e^{-x/5}\left(x^2 - 20x + 50\right)$

(*) $f''(0) = 2 > 0\ldots$

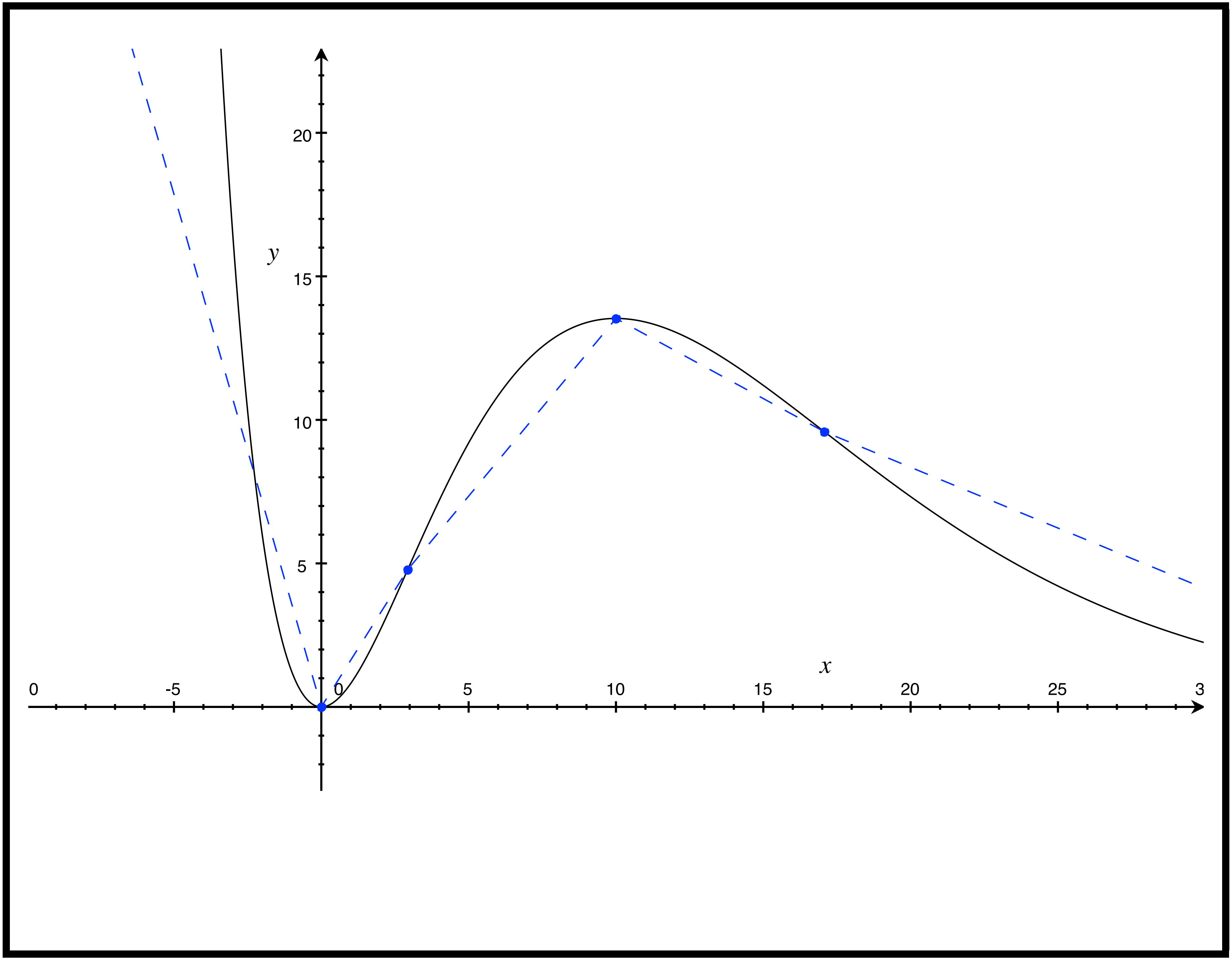
so $y = f(x)$ is concave up in $(-\infty, 2.93)$

(*) $f''(10) = \frac{e^{-2}}{25} \cdot (-50) < 0\ldots$

so $y = f(x)$ is concave down in $(2.93, 17.07)$

(*) $f''(20) = \frac{e^{-4}}{25} \cdot (50) > 0\ldots$

so $y = f(x)$ is concave up in $(17.07, \infty)$



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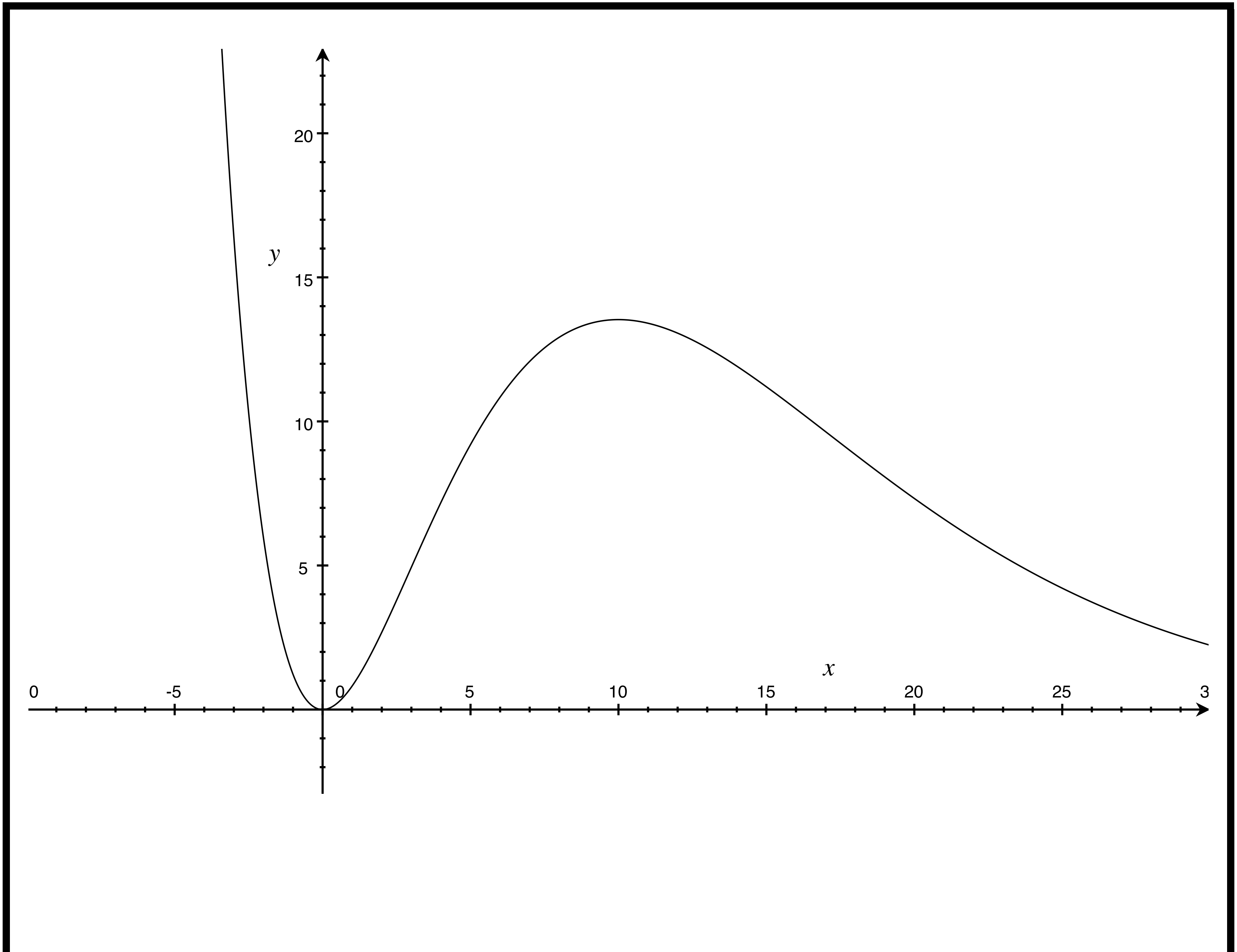

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