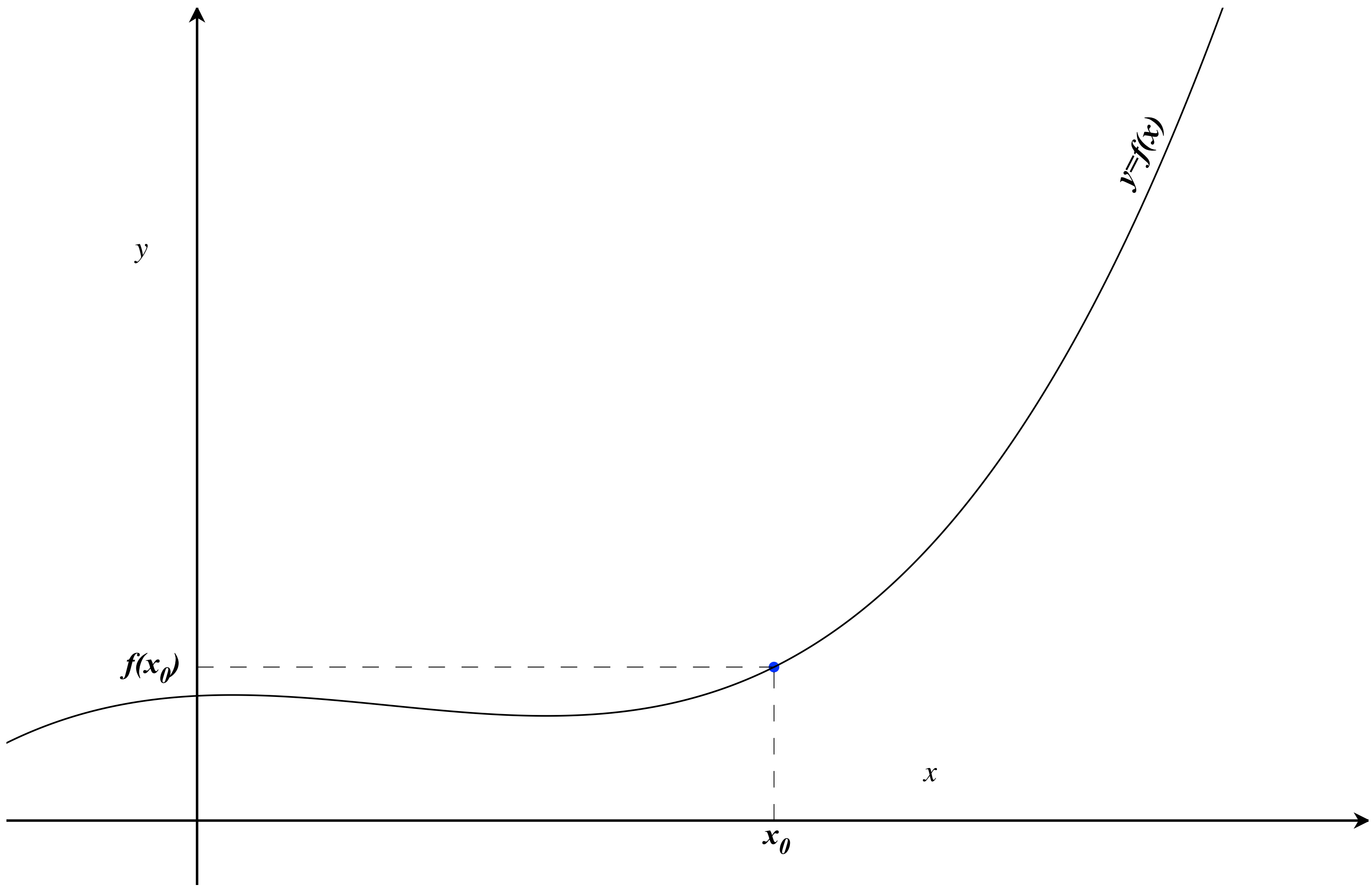
Question: How do we find the slope of the graph $y = f(x)$ at a point $(x_0, f(x_0))$ on the graph?



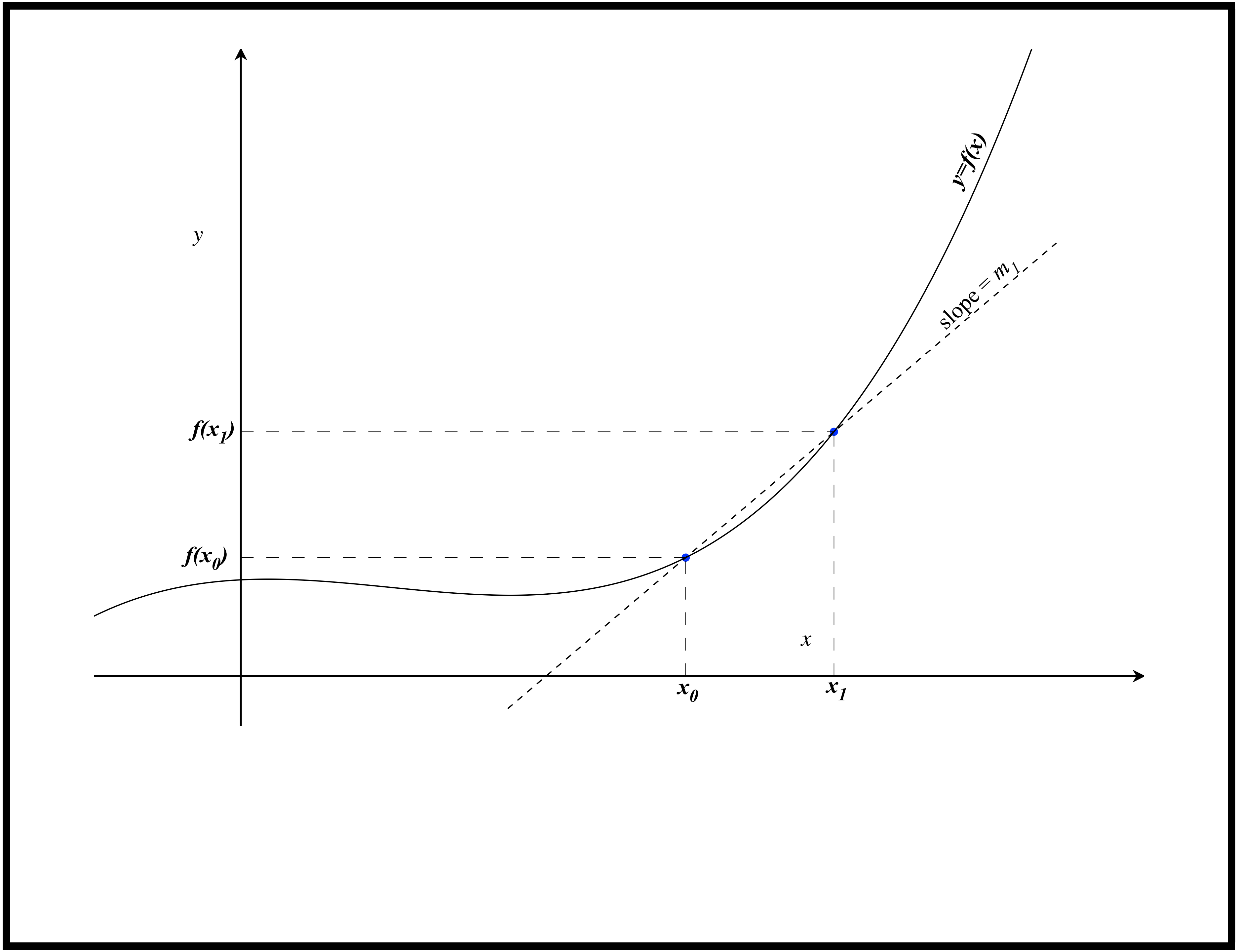
Step 1. *Find an approximation to the slope.*

Problem: we need two points to calculate a slope but we have only one, $(x_0, f(x_0))$.

Solution: choose another point on the graph, say $(x_1, f(x_1))$, and calculate the slope of the (secant) line that connects these two points:

$$m_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

The slope m_1 is an approximation of the (unknown) slope that we seek.

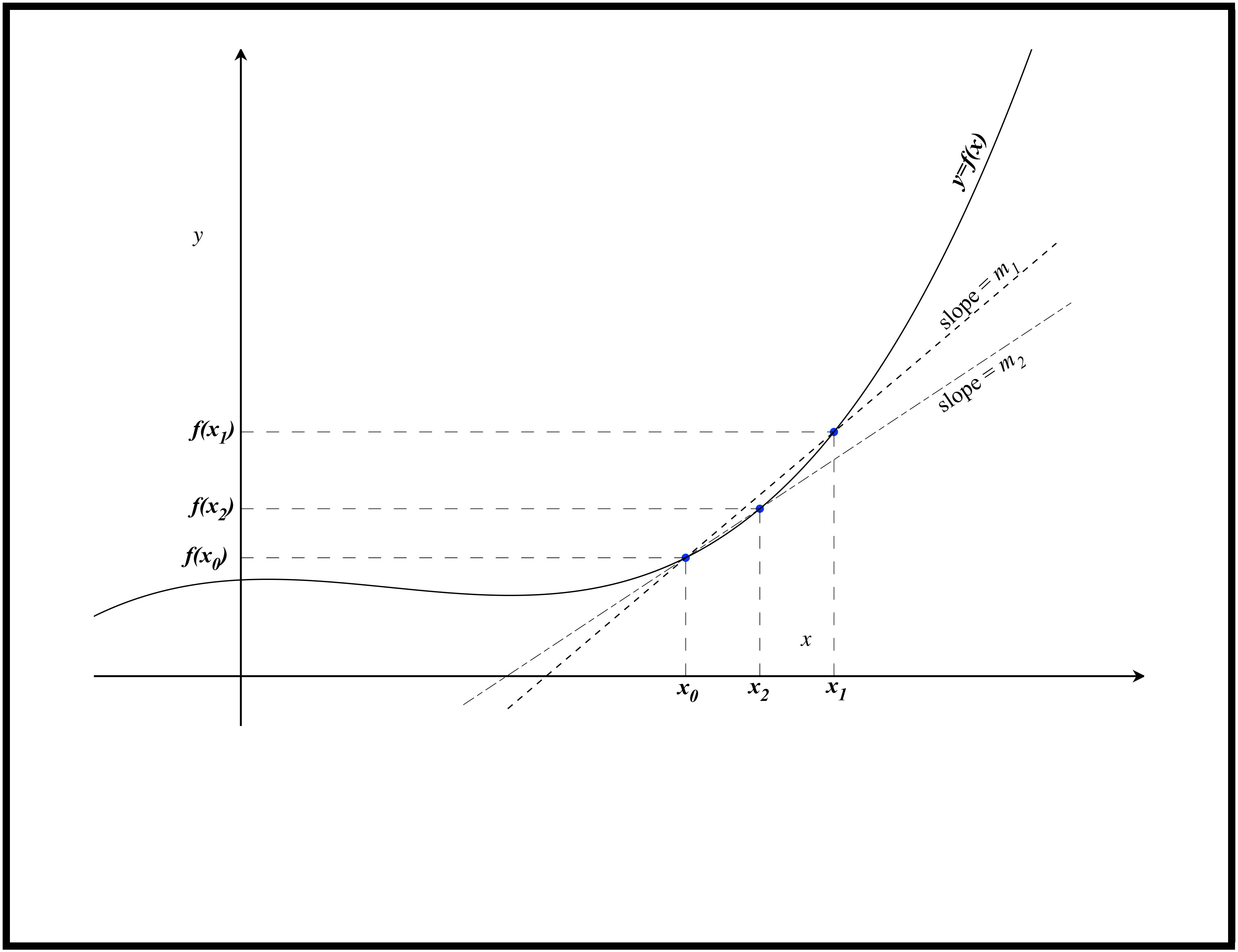

y=f(x)
y
slope = m_1
$f(x_1)$
$f(x_0)$
x
x_0
x_1

Step 2. *Find a (repeatable) way to improve the approximation.*

Intuition: If we choose a point x_2 that is closer to x_0 than x_1, and find the slope

$$m_2 = \frac{f(x_2) - f(x_0)}{x_2 - x_0}$$

of the line connecting $(x_0, f(x_0))$ to $(x_2, f(x_2))$, then m_2 will be a better approximation to the slope ***at*** $(x_0, f(x_0))$ than m_1.



y
y=f(x)
slope = m_1
slope = m_2
$f(x_1)$
$f(x_2)$
$f(x_0)$
x
x_0
x_2
x_1

Step 3 and beyond: *Repeat step 2 and take a limit.*

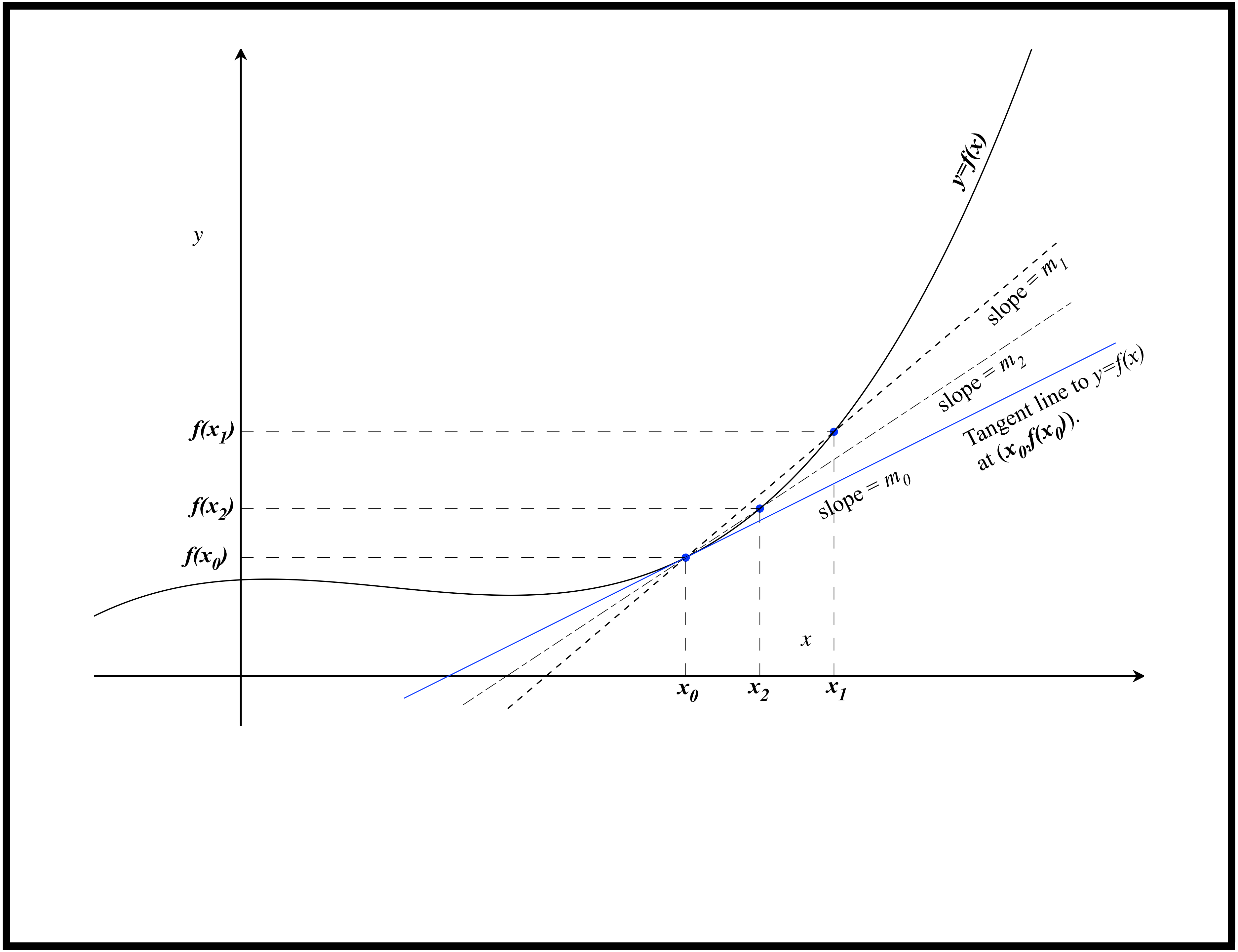
Intuition: If we continue to choose points on the graph that are closer and closer to $(x_0, f(x_0))$ and compute the slopes of the secant lines connecting these points to $(x_0, f(x_0))$, then these slopes should approach the slope ***at*** $(x_0, f(x_0))$, ***if it exists.***

Definition: *The slope of the graph $y = f(x_0)$ at the point $(x_0, f(x_0))$ is given by*

$$m_0 = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

if the limit exists. *If the limit does not exist, then the graph does not have a slope at that point.*

Definition: *If the graph $y = f(x)$ has a slope m_0 at $(x_0, f(x_0))$, then the straight line that passes through $(x_0, f(x_0))$ with slope m_0 is called the **tangent line** to the graph at that point.*



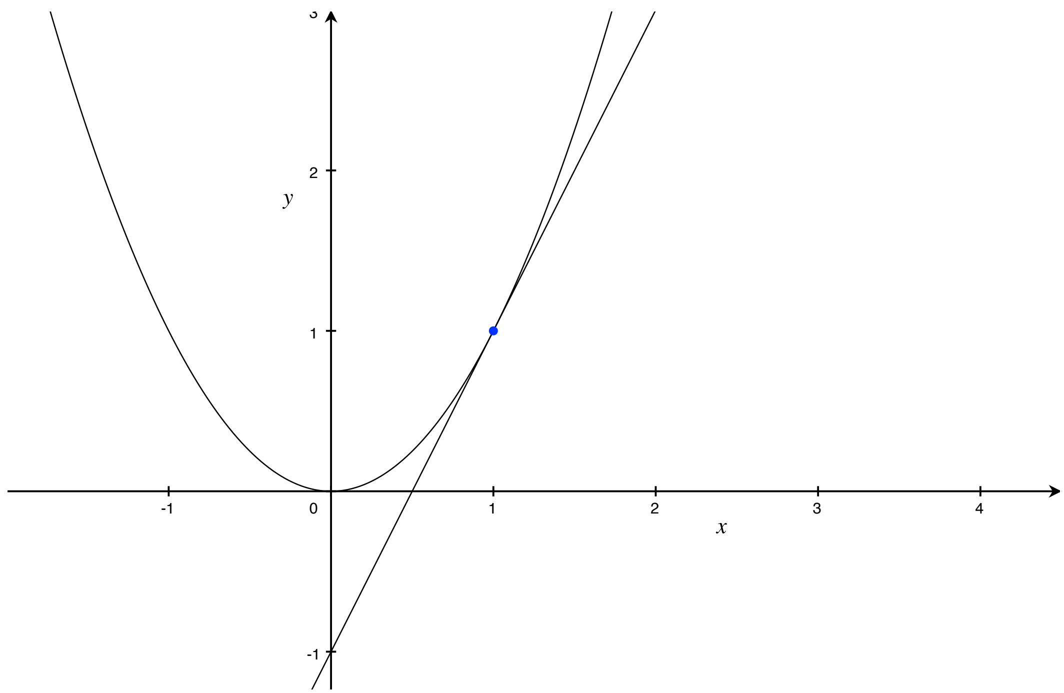
y
y=f(x)
slope = m_1
slope = m_2
Tangent line to $y=f(x)$
at $(x_0,f(x_0))$.
slope = m_0
$f(x_1)$
$f(x_2)$
$f(x_0)$
x
x_0
x_2
x_1

Example 1: Find the slope of the graph $y = x^2$ at the point $(1, 1)$, and find the equation of the tangent line to $y = x^2$ at $(1, 1)$.

$$m = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim_{x \to 1} x + 1 = 2.$$

The tangent line we seek passes through the point $(1, 1)$ with slope $m = 2$... Use the *point-slope* formula:

$$y - 1 = 2(x - 1) \implies y = 2x - 1.$$



Terminology and Notation:

The *slope* of $y = f(x)$ at $(x_0, f(x_0))$ is called the ***derivative*** of $y = f(x)$ at x_0, and denoted by $f'(x_0)$ or $y'(x_0)$. I.e.,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

(if the limit exists).

Using this notation, the equation of the tangent line to $y = f(x)$ at $(x_0, f(x_0))$ is given by

$$y - f(x_0) = f'(x_0)(x - x_0) \quad \text{or} \quad y = f(x_0) + f'(x_0)(x - x_0).$$

Modifying the limit. If we write $x = x_0 + h$, then $x - x_0 = h$, and $x \to x_0$ is the same as $h \to 0$. This means that we can write the definition of the derivative as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Example 1, revisited: The derivative of $y = x^2$ at $x = 1$ is

$$\begin{aligned} y'(1) &= \lim_{h \to 0} \frac{(1+h)^2 - 1}{h} \\ &= \lim_{h \to 0} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(2+h)}{\cancel{h}} = \lim_{h \to 0} 2 + h = 2. \end{aligned}$$

Example 2: Find the derivative of $y = x^2$ at the points $x = 2$ and $x = -1$.

$$\begin{aligned} y'(2) &= \lim_{h\to 0} \frac{(2+h)^2 - 4}{h} \\ &= \lim_{h\to 0} \frac{\not{4} + 4h + h^2 - \not{4}}{h} \\ &= \lim_{h\to 0} \frac{\not{h}(4+h)}{\not{h}} = \lim_{h\to 0} 4 + h = 4. \end{aligned}$$

and

$$\begin{aligned} y'(-1) &= \lim_{h\to 0} \frac{(-1+h)^2 - 1}{h} \\ &= \lim_{h\to 0} \frac{\not{1} - 2h + h^2 - \not{1}}{h} \\ &= \lim_{h\to 0} \frac{\not{h}(-2+h)}{\not{h}} = \lim_{h\to 0} -2 + h = -2. \end{aligned}$$

Definition: The derivative of the function $y = f(x)$ is the function $f'(x)$ defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

at every point x where the limit exists.

Terminology: If $f'(x)$ is defined at a point x_0, then the function $f(x)$ is said to be *differentiable* at x_0. If $f'(x)$ exists for every point x in some interval $I = (a, b)$, then $f(x)$ is differentiable in I.

Example 3: Find the derivative of $f(x) = x^2$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h \to 0} 2x + h = 2x. \end{aligned}$$

This limit exists for all x, so $f(x) = x^2$ is differentiable on the entire real line.