Definition: The derivative of the function $y = f(x)$ is the function $f'(x)$ defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

at every point x where the limit exists.

Terminology: If $f'(x)$ is defined at a point x_0, then the function $f(x)$ is said to be *differentiable* at x_0. If $f'(x)$ exists for every point x in some interval $I = (a, b)$, then $f(x)$ is differentiable in I.

Example 1: Find the derivative of $f(x) = x^2$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h \to 0} 2x + h = 2x. \end{aligned}$$

This limit exists for all x, so $f(x) = x^2$ is differentiable on the entire real line.

Example 2. Find the derivative of the function $l(x) = ax + b$.

$$l'(x) = \lim_{h \to 0} \frac{a(x+h) + b - (ax+b)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{ax} + ah + \cancel{b} - \cancel{ax} - \cancel{b})}{h}$$

$$= \lim_{h \to 0} \frac{a\cancel{h}}{\cancel{h}} = \lim_{h \to 0} a = a$$

Observation: The graph $y = l(x)$ is a straight line with slope a, so the fact that $l'(x) = a$ shows that the derivative, as defined, extends the concept of slope from straight lines to more general curves.

Comment: We don't have to use h to denote the change in x. Another (perhaps better) symbol for the change in x is Δx, pronounced *delta x*.

Example 3: Find the derivative of $g(x) = x^{1/2}$, and determine the interval(s) where the function is differentiable.

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^{1/2} - x^{1/2}}{\Delta x}$$

$$= \ldots ?$$

$$= \lim_{\Delta x \to 0} \left[\left(\frac{(x+\Delta x)^{1/2} - x^{1/2}}{\Delta x} \right) \cdot \overbrace{\left(\frac{(x+\Delta x)^{1/2} + x^{1/2}}{(x+\Delta x)^{1/2} + x^{1/2}} \right)}^{=1} \right]$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x} + \Delta x - \cancel{x}}{\Delta x \left((x+\Delta x)^{1/2} + x^{1/2} \right)} = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x} \left((x+\Delta x)^{1/2} + x^{1/2} \right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{(x+\Delta x)^{1/2} + x^{1/2}} = \frac{1}{2x^{1/2}}$$

So, the derivative of $g(x) = x^{1/2}$ is $g'(x) = \dfrac{1}{2x^{1/2}}$, which is defined for all $x > 0$. This means that $g(x) = x^{1/2}$ is differentiable in the interval $(0, \infty)$.

Example 4: Find the derivative of $f(x) = 1/x^2$ and determine the intervals where this function is differentiable.

$$f'(x) = \lim_{\Delta x \to 0} \frac{\dfrac{1}{(x+\Delta x)^2} - \dfrac{1}{x^2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{x^2 - (x+\Delta x)^2}{x^2(x+\Delta x)^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^2} - (\cancel{x^2} + 2x\Delta x + (\Delta x)^2)}{x^2(x+\Delta x)^2 \Delta x} = \lim_{\Delta x \to 0} \frac{-2x\cancel{\Delta x} - (\Delta x)^{\cancel{2}}}{x^2(x+\Delta x)^2 \cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{-2\cancel{x}}{x^{\cancel{2}}(x+\Delta x)^2} - \frac{\Delta x}{x^2(x+\Delta x)^2}$$

$$= \lim_{\Delta x \to 0} -\frac{2}{x(x+\Delta x)^2} - \lim_{\Delta x \to 0} \frac{\Delta x}{x^2(x+\Delta x)^2} = -\frac{2}{x^3} - 0 = -\frac{2}{x^3}$$

This means that $f(x) = 1/x^2$ is differentiable in $(-\infty, 0)$ and $(0, \infty)$.

Let's collect the results from the examples thus far...

- $f(x) = x^2 \implies f'(x) = 2x \qquad \implies \left(x^2\right)' = 2 \cdot x^1$
- $l(x) = x \implies l'(x) = 1 \qquad \implies \left(x^1\right)' = 1 \cdot x^0$
- $g(x) = x^{1/2} \implies g'(x) = \dfrac{1}{2x^{1/2}} \qquad \implies \left(x^{1/2}\right)' = \dfrac{1}{2}x^{-1/2}$
- $f(x) = \dfrac{1}{x^2} \implies f'(x) = -\dfrac{2}{x^3} \qquad \implies \left(x^{-2}\right)' = -2 \cdot x^{-3}$

Observation: The derivatives of these functions all satisfy the rule

$$\left(x^k\right)' = kx^{k-1}.$$

Notation: It is common to use the notation $\dfrac{d}{dx}$ for the operation of differentiation—finding a derivative. I.e.,

$$\frac{d}{dx}f(x) = f'(x).$$

Power rule: If k is any real number, then

$$\frac{d}{dx}\left(x^k\right) = kx^{k-1}.$$

Comment: This rule is relatively easy to justify if k is a positive integer (as done in the book). It is not much harder to do if k is a negative integer. When k is a *rational* number the justification is more challenging, and the most challenging case is when k is irrational.

More basic rules...

Constant functions:

$$\frac{d}{dx}C = 0$$

Because

$$\frac{d}{dx}C = \lim_{h\to 0}\frac{C-C}{h} = \lim_{h\to 0}\frac{0}{h} = \lim_{h\to 0} 0 = 0.$$

(*) This agrees with the fact that the graph of a constant function is a horizontal line, and the slope of a horizontal line is 0.

Sums and differences:

$$\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x).$$

Because (for sums, differences are handled analogously)

$$\begin{aligned}
\frac{d}{dx}(f(x)+g(x)) &= \lim_{h\to 0} \frac{(f(x+h)+g(x+h)) - (f(x)+g(x))}{h} \\
&= \lim_{h\to 0} \frac{(f(x+h)-f(x)) + (g(x+h)-g(x))}{h} \\
&= \lim_{h\to 0} \left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h))-g(x)}{h} \right) \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h))-g(x)}{h} \\
&= f'(x) + g'(x)
\end{aligned}$$

Example 5:

$$\frac{d}{dx}\left(x^2 + \frac{1}{x}\right) = \frac{d}{dx}\left(x^2 + x^{-1}\right) = 2x + (-1)\cdot x^{-2} = 2x - x^{-2}.$$

Constant multiples:

$$\frac{d}{dx}(Cf(x)) = Cf'(x).$$

Because

$$\begin{aligned}\frac{d}{dx}(Cf(x)) &= \lim_{h\to 0}\frac{Cf(x+h)-Cf(x)}{h}\\ &= \lim_{h\to 0} C\left(\frac{f(x+h)-f(x)}{h}\right)\\ &= C\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = Cf'(x)\end{aligned}$$

Example 6: Find the derivative of $y = 2x^3 - 3x^2 + 4x - 5$.

$$\begin{aligned}y' &= \frac{d}{dx}(2x^3) - \frac{d}{dx}(3x^2) + \frac{d}{dx}(4x) - \frac{d}{dx}(5)\\ &= 2\frac{d}{dx}(x^3) - 3\frac{d}{dx}(x^2) + 4\frac{d}{dx}(x) - \frac{d}{dx}(5)\\ &= 2(3x^2) - 3(2x) + 4(1) - 0 = 6x^2 - 6x + 4\end{aligned}$$

Example 7: Find the derivative of $f(x) = \dfrac{3x^2 - 1}{5\sqrt{x}}$.

First, simplify the $f(x)$ and rewrite it as a sum of multiples of powers

$$f(x) = \frac{3x^2 - 1}{5\sqrt{x}} = \frac{3x^2}{5x^{1/2}} - \frac{1}{5x^{1/2}} = \frac{3}{5}x^{3/2} - \frac{1}{5}x^{-1/2}$$

Then differentiate

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{3}{5}x^{3/2}\right) - \frac{d}{dx}\left(\frac{1}{5}x^{-1/2}\right) \\ &= \frac{3}{5} \cdot \frac{d}{dx}\left(x^{3/2}\right) - \frac{1}{5} \cdot \frac{d}{dx}\left(x^{-1/2}\right) \\ &= \frac{3}{5} \cdot \frac{3}{2}x^{1/2} - \frac{1}{5} \cdot \left(-\frac{1}{2}\right)x^{-3/2} \\ &= \frac{9}{10}x^{1/2} + \frac{1}{10}x^{-3/2} \end{aligned}$$