Average elasticity: The average elasticity of one variable with respect to another is the ratio of their respective *percentage* changes.

Example: If income increases by 5% over a certain period and as result, consumption increases by 4.2%, then the average *income-elasticity* of *consumption* is

$$E_{C/I} = \frac{\%\Delta C}{\%\Delta I} = \frac{4.2\%}{5\%} = 0.84.$$

Definition: If $y = f(x)$, then the average elasticity of y with respect to x (also called the x-elasticity of y) over the interval $(x, x + \Delta x)$ is

$$E_{y/x} = \frac{\%\Delta y}{\%\Delta x} = \frac{\dfrac{\Delta y}{y} \cdot 100\%}{\dfrac{\Delta x}{x} \cdot 100\%} = \frac{\left(\dfrac{f(x+\Delta x) - f(x)}{f(x)}\right) \cdot 100\%}{\dfrac{\Delta x}{x} \cdot 100\%}$$

Observation: The elasticity is *units-free*—a pure number—because the units for x cancel each other out in the denominator and the units for y cancel in the numerator.

The formula for elasticity simplifies:

$$E_{y/x} = \frac{\%\Delta y}{\%\Delta x} = \frac{\dfrac{\Delta y}{y_0} \cdot \cancel{100\%}}{\dfrac{\Delta x}{x_0} \cdot \cancel{100\%}} = \frac{\Delta y}{\Delta x} \cdot \frac{x}{y}.$$

Example: Suppose that $y = 0.3x^2$. What is $E_{y/x}$ in the interval $(1, 3)$?

In this case $x = 1$ and $\Delta x = 3 - 1 = 2$, and $y = y(1) = 0.3$ and $\Delta y = y(3) - y(1) = 2.7 - 0.3 = 2.4$. Therefore

$$E_{y/x} = \frac{2.4}{2} \cdot \frac{1}{0.3} = 4.$$

By what percentage does y increase in this interval? Two ways to answer...

$$\%\Delta y = \frac{\Delta y}{y} \cdot 100\% = \frac{2.4}{0.3} \cdot 100\% = 800\%,$$

or

$$\%\Delta y = E_{y/x} \cdot \%\Delta x = 4 \cdot \frac{\Delta x}{x} \cdot 100\% = 4 \cdot 200\% = 800\%.$$

Definition: The ***point-elasticity*** of y with respect to x is the *limit of the average elasticity as* $\Delta x \to 0$. We denote point-elasticity by the Greek letter η (eta), so

$$\eta_{y/x} = \lim_{\Delta x \to 0} E_{y/x} = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} \cdot \frac{x}{y} \right) = \left(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \right) \cdot \frac{x}{y} = \frac{dy}{dx} \cdot \frac{x}{y}.$$

Observation: As defined, if $y = f(x)$, then $\eta_{y/x}$ is also a function of x:

$$\eta_{y/x} = \frac{dy}{dx} \cdot \frac{x}{y} = \frac{x f'(x)}{f(x)}.$$

Example: Suppose that $y = 2x^{0.4}$, then

$$\eta_{y/x} = 0.8x^{-0.6} \cdot \frac{x}{2x^{0.4}} = \frac{0.8}{2} x^{-0.6} \cdot x^{0.6} = 0.4.$$

In this case, elasticity is *constant*.

Price-elasticity of demand.

Example: If $q = 300 - 1.2p$ is the demand function for a firm, then

$$\eta_{q/p} = (300 - 1.2p)' \cdot \frac{p}{300 - 1.2p} = -1.2 \cdot \frac{p}{300 - 1.2p} = -\frac{1.2p}{300 - 1.2p}$$

is the price-elasticity of demand for the firm's product, *expressed as a function of* p (and in this case, elasticity is not constant).

What is the price-elasticity of demand when the price is $p = 200$?

$$\eta_{q/p}\Big|_{p=200} = -\frac{240}{300 - 240} = -4.$$

Question: Demand equations are often written in the form $p = f(q)$, with the demand q as the independent variable and the price p as the dependent variable.

How can we compute the price-elasticity of demand in such cases?

Answer: we return to the definition...

$$\eta_{q/p} = \lim_{\Delta p \to 0} \frac{\Delta q}{\Delta p} \cdot \frac{p}{q}$$

and make two observations...

1. If $\Delta p \to 0$ then $\Delta q \to 0$ and *vice versa*, and

2. $\dfrac{\Delta q}{\Delta p} = \dfrac{1}{\frac{\Delta p}{\Delta q}}$, so

$$\lim_{\Delta p \to 0} \frac{\Delta q}{\Delta p} = \lim_{\Delta q \to 0} \frac{1}{\dfrac{\Delta p}{\Delta q}} = \frac{1}{\lim_{\Delta q \to 0} \dfrac{\Delta p}{\Delta q}} = \frac{1}{dp/dq}.$$

Plugging this back into the definition of elasticity gives

$$\eta_{q/p} = \lim_{\Delta p \to 0} \frac{\Delta q}{\Delta p} \cdot \frac{p}{q} = \frac{1}{dp/dq} \cdot \frac{p}{q} = \frac{\dfrac{p}{q}}{\dfrac{dp}{dq}}$$

Example: Suppose that the demand equation for a firm's product is given by $p = 250 - \frac{5}{6}q$. What is the price-elasticity of demand for the firm's product when $p = 200$?

In this case we first find the elasticity as a function of q...

$$\eta_{q/p} = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{250 - \frac{5}{6}q}{q}}{-\frac{5}{6}} = -\frac{300 - q}{q}$$

and then find q when $p = 200$...

$$p = 200 \implies 200 = 250 - \frac{5}{6}q \implies \frac{5}{6}q = 50 \implies q = 60$$

and then compute the elasticity

$$\eta_{q/p}\Big|_{p=200} = \eta_{q/p}\Big|_{q=60} = -\frac{300 - 60}{60} = -4$$