Differentiating logarithm functions.

To differentiate $y = \ln x$, we must return to the definition...

$$\frac{d}{dx}(\ln x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h}$$

and simplify, using algebraic properties of $\ln x$...

$$= \lim_{h\to 0} \frac{1}{h}\left(\ln(x+h) - \ln x\right)$$

$$= \lim_{h\to 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right)$$

$$= \lim_{h\to 0} \ln\left[\left(1 + \frac{h}{x}\right)^{1/h}\right]$$

and use the *continuity* of $\ln x$...

$$= \ln\left[\lim_{h\to 0}\left(1 + \frac{h}{x}\right)^{1/h}\right]$$

What next?

Remember the special limit

$$\lim_{u \to 0} (1+u)^{1/u} = e \ldots$$

and substitute (rename)

$$\frac{h}{x} = u \implies h = ux \implies \frac{1}{h} = \frac{1}{ux} = \frac{1}{u} \cdot \frac{1}{x} \ldots$$

So

$$\lim_{h \to 0} \left(1 + \frac{h}{x}\right)^{1/h} = \lim_{u \to 0} (1+u)^{\frac{1}{u} \cdot \frac{1}{x}}$$

$$= \lim_{u \to 0} \left((1+u)^{\frac{1}{u}}\right)^{\frac{1}{x}}$$

$$= \left(\lim_{u \to 0} (1+u)^{\frac{1}{u}}\right)^{\frac{1}{x}}$$

(because $f(x) = a^{1/x}$ is continuous)

$$= e^{1/x}$$

Returning to $\frac{d}{dx}(\ln x)$...

$$\begin{aligned}
\frac{d}{dx}(\ln x) &= \lim_{h\to 0} \frac{\ln(x+h)-\ln x}{h} \\
&\ \vdots \\
&= \ln\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{1/h}\right] \\
&= \ln(e^{1/x}) \\
&= \frac{1}{x}\ .
\end{aligned}$$

I.e.,

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

Example 1.

$$\frac{d}{dx}\left(3x^2 \ln x\right) = 6x \ln x + 3x^2 \cdot \frac{1}{x} = 6x \ln x + 3x$$

(product rule)

Example 2.

$$\frac{d}{dx} \ln\left(5x^2 + 3x + 1\right) = \frac{1}{5x^2 + 3x + 1} \cdot (10x + 3) = \frac{10x + 3}{5x^2 + 3x + 1}$$

(chain rule)

Example 3. Differentiate $y = \ln\left(5x^2\right)$.

We can use the chain rule again:

$$y' = \frac{1}{5x^2} \cdot 10x = \frac{10x}{5x^2} = \frac{2}{x} \ldots$$

or we can simplify and then differentiate:

$$y = \ln\left(5x^2\right) = \ln 5 + \ln x^2 = \ln 5 + 2 \ln x \implies y' = 0 + 2 \cdot \frac{1}{x} = \frac{2}{x}$$

Example 4. Find the equation of the tangent line to the graph

$$y = \ln\left(\frac{x^2 + x - 1}{4x - 3}\right)$$

at the point where $x = 1$.

Point: $y(1) = \ln\left(\frac{1 + 1 - 1}{4 - 3}\right) = \ln 1 = 0$, so the point is $(1, 0)$.

Slope: We need to compute $y'(1)$, and once again, it is easier to simplify before we differentiate:

$$y = \ln\left(\frac{x^2 + x - 1}{4x - 3}\right) = \ln(x^2 + x - 1) - \ln(4x - 3)$$

$$\implies y' = \frac{2x + 1}{x^2 + x - 1} - \frac{4}{4x - 3}$$

$$\implies y'(1) = \frac{3}{1} - \frac{4}{1} = -1.$$

So the equation of the tangent line is

$$y = y(1) + y'(1)(x - 1) \implies y = -(x - 1) \implies y = 1 - x.$$

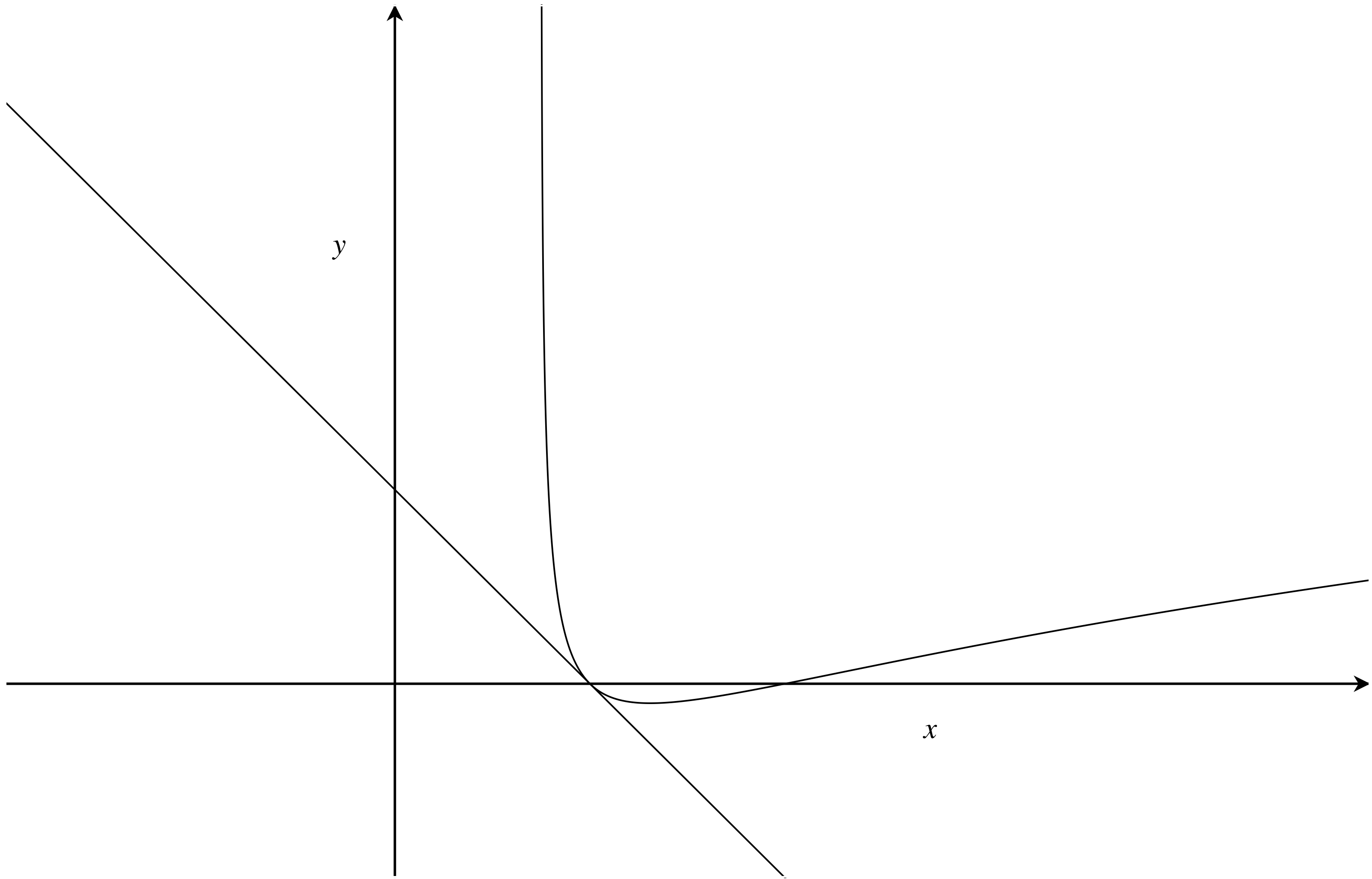

Figure 1: Graph of $y = \ln\left(\frac{x^2+x-1}{4x-3}\right)$ and its tangent line at $(1, 0)$

Logarithmic differentiation:

If $f(x)$ is any (differentiable) function, then

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}.$$

This is called the *logarithmic derivative* of $f(x)$.

In some cases, it is easier to compute the $\frac{d}{dx}\ln(f(x))$ than it is to compute $f'(x)$, because $\ln(f(x))$ is a simpler function than $f(x)$ itself.

We can take advantage of this to find $f'(x)$ because

$$f'(x) = f(x) \cdot \left(\frac{d}{dx}\ln(f(x))\right).$$

This is called logarithmic differentiation.

We will use this idea in the following special, but **important** case...

Differentiating the exponential function.

Observation: $\ln(e^x) = x$ which is a simpler function (to differentiate) than e^x. So we will use logarithmic differentiation to find $\frac{d}{dx}e^x$:

$$\frac{d}{dx}e^x = e^x \cdot \left(\frac{d}{dx}(\ln(e^x))\right) = e^x \cdot \left(\frac{d}{dx}x\right) = e^x \cdot 1 = e^x.$$

I.e.,

$$\boxed{\frac{d}{dx}e^x = e^x}$$

Example 5.

$$\frac{d}{dx}\left(3e^x x^{1/2}\right) = 3e^x x^{1/2} + 3e^x \cdot \frac{1}{2}x^{-1/2} = 3e^x x^{1/2} + \frac{3}{2}e^x x^{-1/2}$$

product rule.

Example 6.

$$\left(e^{x^2-3x+1}\right)' = e^{x^2-3x+1}(2x-3)$$

chain rule

Example 7. Find the marginal revenue function for the firm whose demand equation is given by

$$p = 15e^{2-0.05q}.$$

First, find the revenue function

$$r = pq = 15qe^{2-0.05q}.$$

Now differentiate, using the product and chain rules:

$$\frac{dr}{dq} = 15e^{2-0.05q} + 15qe^{2-0.05q}(-0.05) = (15 - 0.75q)e^{2-0.05q}.$$

The consumption function for a small country is given by

$$C = \ln\left(\frac{e^{0.95Y}}{e^{0.2Y}+5}\right),$$

where Y is national income, measured in \$billions.

(a) How much is consumed when $Y = 10$?

$$C(10) = \ln\left(\frac{e^{9.5}}{e^{2}+5}\right) \approx 6.983$$

(b) What is the marginal propensity to consume when $Y = 10$?

$$\begin{aligned}
\frac{dC}{dY} &= \frac{d}{dY}\left(\ln\left(\frac{e^{0.95Y}}{e^{0.2Y}+5}\right)\right) \\
&= \frac{d}{dY}\left(\ln\left(e^{0.95Y}\right) - \ln\left(e^{0.2Y}+5\right)\right) \\
&= \frac{d}{dY}\left(0.95Y - \ln\left(e^{0.2Y}+5\right)\right) = 0.95 - \frac{0.2e^{0.2Y}}{e^{0.2Y}+5}
\end{aligned}$$

$$\left.\frac{dC}{dY}\right|_{Y=10} = 0.95 - \frac{0.2e^{2}}{e^{2}+5} \approx 0.8307$$

(c) Compute the limit $\lim_{Y\to\infty} \frac{dC}{dY}$, and interpret the result.

$$\lim_{Y\to\infty} \frac{dC}{dY} = \lim_{Y\to\infty} \left(0.95 - \frac{0.2e^{0.2Y}}{e^{0.2Y}+5}\right)$$

$$= 0.95 - \lim_{Y\to\infty} \frac{0.2e^{0.2Y} \cdot e^{-0.2Y}}{(e^{0.2Y}+5)\cdot e^{-0.2Y}}$$

$$= 0.95 - \lim_{Y\to\infty} \frac{0.2}{1+5e^{-0.2Y}} = 0.95 - \frac{0.2}{1} = 0.75$$

Interpretation: *When income grows large, the nation will tend to consume \$0.75 of each additional dollar of income.*