The product rule.

$$(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x).$$

The quotient rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Example 1. $\dfrac{d}{dx}\left((x^2+4x+5)\sqrt[3]{x}\right) = \dots$

$$\begin{aligned} = \frac{d}{dx}\left((x^2+4x+5)x^{1/3}\right) &= (x^2+4x+5)' \cdot x^{1/3} + (x^2+4x+5)\left(x^{1/3}\right)' \\ &= (2x+4)x^{1/3} + (x^2+4x+5) \cdot \frac{1}{3}x^{-2/3} \\ &= 2x^{4/3} + 4x^{1/3} + \frac{1}{3}x^{4/3} + \frac{4}{3}x^{1/3} + \frac{5}{3}x^{-2/3} \\ &= \frac{7}{3}x^{4/3} + \frac{16}{3}x^{1/3} + \frac{5}{3}x^{-2/3} \end{aligned}$$

Check: $(x^2+4x+5)\sqrt[3]{x} = x^{7/3}+4x^{4/3}+5x^{1/3}$, so

$$\left((x^2+4x+5)\sqrt[3]{x}\right)' = \left(x^{7/3}+4x^{4/3}+5x^{1/3}\right)' = \frac{7}{3}x^{4/3}+\frac{16}{3}x^{1/3}+\frac{5}{3}x^{-2/3}$$

Example 2. Find the interval(s) where the slope of $s = \dfrac{3t}{t^2+1}$ is positive.

The slope of this graph is positive at the points t where $ds/dt > 0$, and...

$$\begin{aligned}\frac{ds}{dt} = \left(\frac{3t}{t^2+1}\right)' &= \frac{(3t)'(t^2+1) - 3t(t^2+1)'}{(t^2+1)^2} \\ &= \frac{3(t^2+1) - 3t\cdot 2t}{(t^2+1)^2} \\ &= \frac{3-3t^2}{(t^2+1)^2} = \frac{3(1-t^2)}{(t^2+1)^2}.\end{aligned}$$

Since $3 > 0$ and $(t^2+1)^2 > 0$ for all t, it follows that $ds/dt > 0$ when $1-t^2 > 0$. I.e., the slope is positive when $-1 < t < 1$.

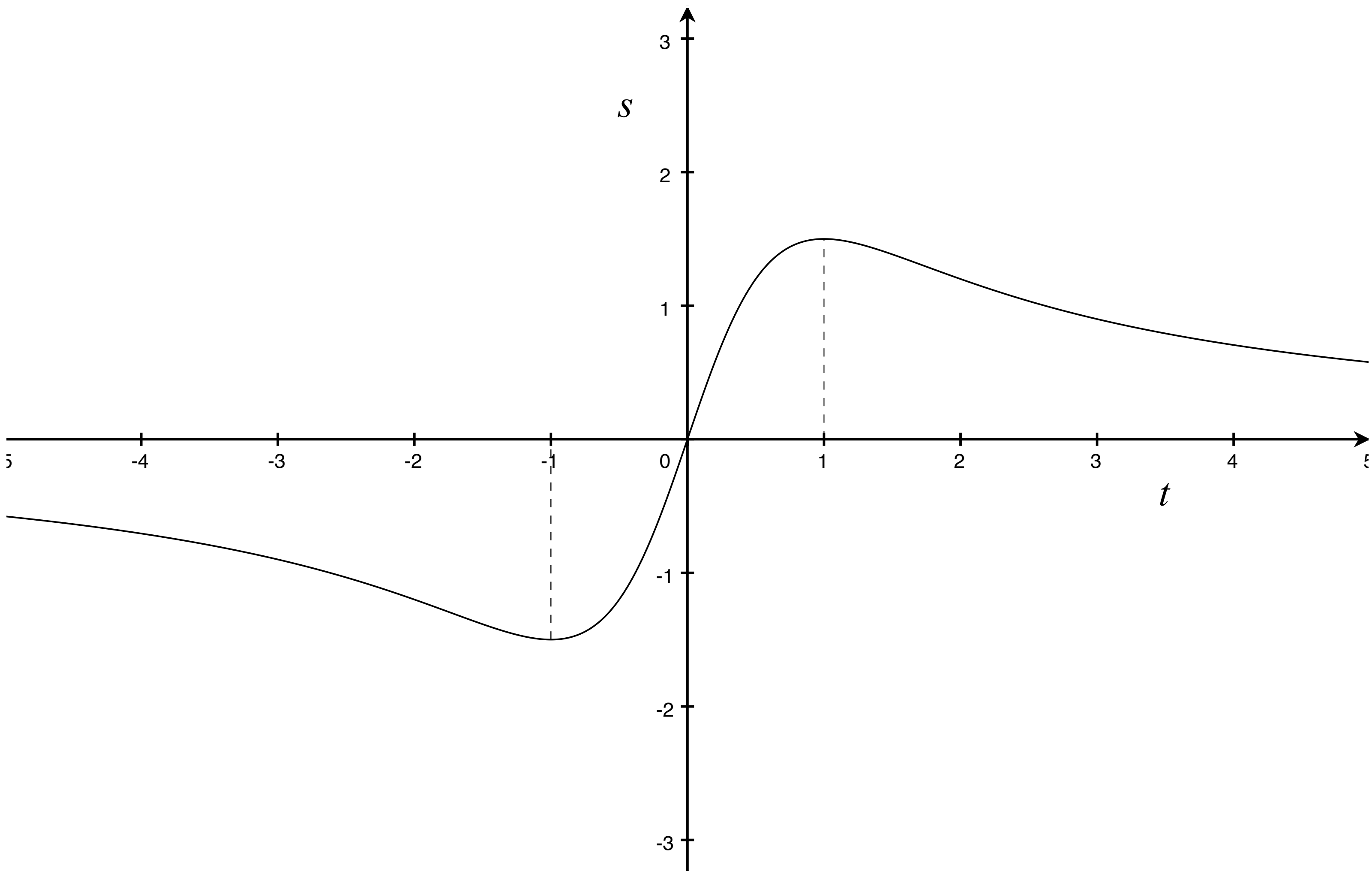

Figure 1: The graph of $s = \dfrac{3t}{t^2+1}$.

The product rule.

$$(f(x)\cdot g(x))' = f'(x)g(x) + f(x)g'(x).$$

Because...

$$\begin{aligned}
(f(x)\cdot g(x))' &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)\overbrace{-f(x)g(x+h)+f(x)g(x+h)}^{=0} - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} \\
&\qquad + \lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} g(x+h) + \lim_{h\to 0} f(x)\cdot \frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + f(x)\cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}$$

The quotient rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Because...

$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \lim_{h\to 0} \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{f(x+h)g(x) \overbrace{- f(x)g(x) + f(x)g(x)}^{=0} - f(x)g(x+h)}{g(x)g(x+h)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{f(x+h)g(x) - f(x)g(x)}{g(x)g(x+h)}\right) \\
&\qquad - \lim_{h\to 0} \frac{1}{h}\left(\frac{f(x)g(x+h) - f(x)g(x)}{g(x)g(x+h)}\right)
\end{aligned}$$

$$
\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x+h)g(x)-f(x)g(x)}{g(x)g(x+h)}\right) \\
&\qquad - \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x)g(x+h)-f(x)g(x)}{g(x)g(x+h)}\right) \\
&= \lim_{h\to 0}\frac{1}{g(x)g(x+h)}\left(\frac{f(x+h)g(x)-f(x)g(x)}{h}\right) \\
&\qquad - \lim_{h\to 0}\frac{1}{g(x)g(x+h)}\left(\frac{f(x)g(x+h)-f(x)g(x)}{h}\right) \\
&= \lim_{h\to 0}\frac{1}{g(x)g(x+h)}\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)g(x) \\
&\qquad - \lim_{h\to 0}\frac{1}{g(x)g(x+h)}\lim_{h\to 0}f(x)\left(\frac{g(x+h)-g(x)}{h}\right) \\
&= \frac{1}{g(x)^2}\big(f'(x)g(x)-f(x)g'(x)\big) \\
&= \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}
\end{aligned}
$$

Example. The consumption function of a small nation is given by

$$C = \frac{9Y^2 + 5Y + 100}{10Y + 1},$$

where both annual consumption C and annual income Y are measured in \$ billions.

1. Find the *marginal propensity to consume* and the *marginal propensity to save* when national income is \$8 billion.

Marginal propensity to consume: differentiate...

$$\frac{dC}{dY} = \frac{(18Y + 5)(10Y + 1) - 10(9Y^2 + 5Y + 100)}{(10Y + 1)^2} = \frac{90Y^2 + 18Y - 995}{(10Y + 1)^2}$$

Then evaluate:

$$\left.\frac{dC}{dY}\right|_{Y=8} = \frac{90 \cdot 64 + 18 \cdot 8 - 995}{81^2} \approx 0.7402$$

Marginal propensity to save...?

Accounting identity: $C + S = Y \implies \frac{dC}{dY} + \frac{dS}{dY} = 1 \implies \frac{dS}{dY} = 1 - \frac{dC}{dY}$

So: $\left.\frac{dS}{dY}\right|_{Y=8} = 1 - \left.\frac{dC}{dY}\right|_{Y=8} \approx 1 - 0.7402 = 0.2598$

2. What happens to the MPC as income continues to grow? What does this say about national consumption when income is large?

... *'as Y continues to grow...* $\Longrightarrow \lim_{Y\to\infty}$:

$$\lim_{Y\to\infty} \frac{dC}{dY} = \lim_{Y\to\infty} \frac{90Y^2 + 18Y - 995}{(10Y+1)^2} = \lim_{Y\to\infty} \frac{90Y^2 + 18Y - 995}{100Y^2 + 20Y + 1} = 0.9$$

Now:

(i) If ΔY is small, then $\Delta C \approx \dfrac{dC}{dY} \cdot \Delta Y$.

(ii) If Y is large, then $\frac{dC}{dY} \approx 0.9$.

(iii) If Y grows by \$1, then $\Delta Y = \dfrac{1}{10^9}$ (quite small).

Taken together, (i), (ii) and (iii) lead to the interpretation:

> *When income is large, the nation will tend to consume about* \$0.90 *of each additional dollar of income.*