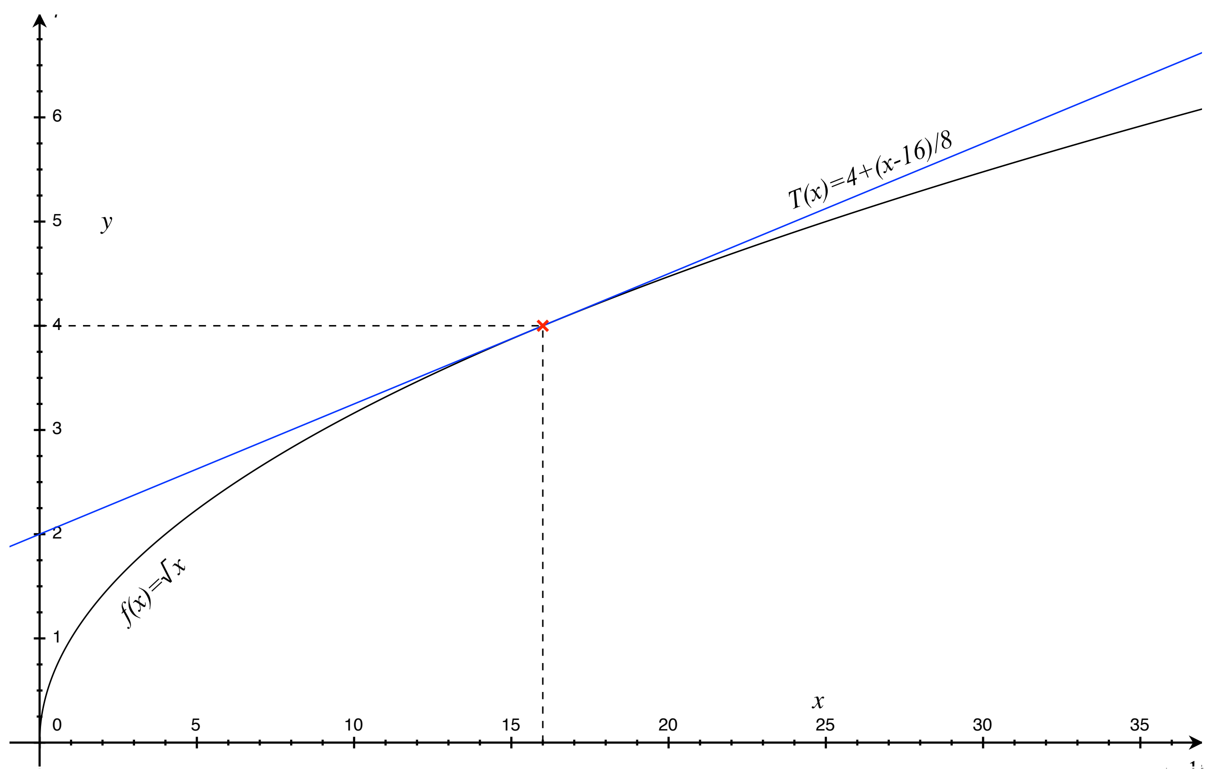
Linear approximation for the function $f(x) = \sqrt{x}$ in the vicinity of the point $x_0 = 16$:

$$\begin{aligned}\sqrt{x} = f(x) &\approx f(x_0) + f'(x_0)(x - x_0) \\ &= (16)^{1/2} + \frac{1}{2}(16)^{-1/2}(x - 16) \\ &= 4 + \frac{1}{8}(x - 16) = T(x)\end{aligned}$$



Observations:

1. $T(16) = 4 = \sqrt{16}$ and $T'(16) = \dfrac{1}{8} = \dfrac{d}{dx}\left(\sqrt{x}\right)\Big|_{x=16}$.

2. The approximation is fairly accurate when x is within 1 or 2 of 16.

3. The approximation becomes increasingly less accurate as x moves away from 16 because...

4. ... the slope of $f(x) = \sqrt{x}$ is changing but the slope of $T(x)$ is not.

To obtain a better approximation, we can try to find a quadratic function $Q(x)$, satisfying

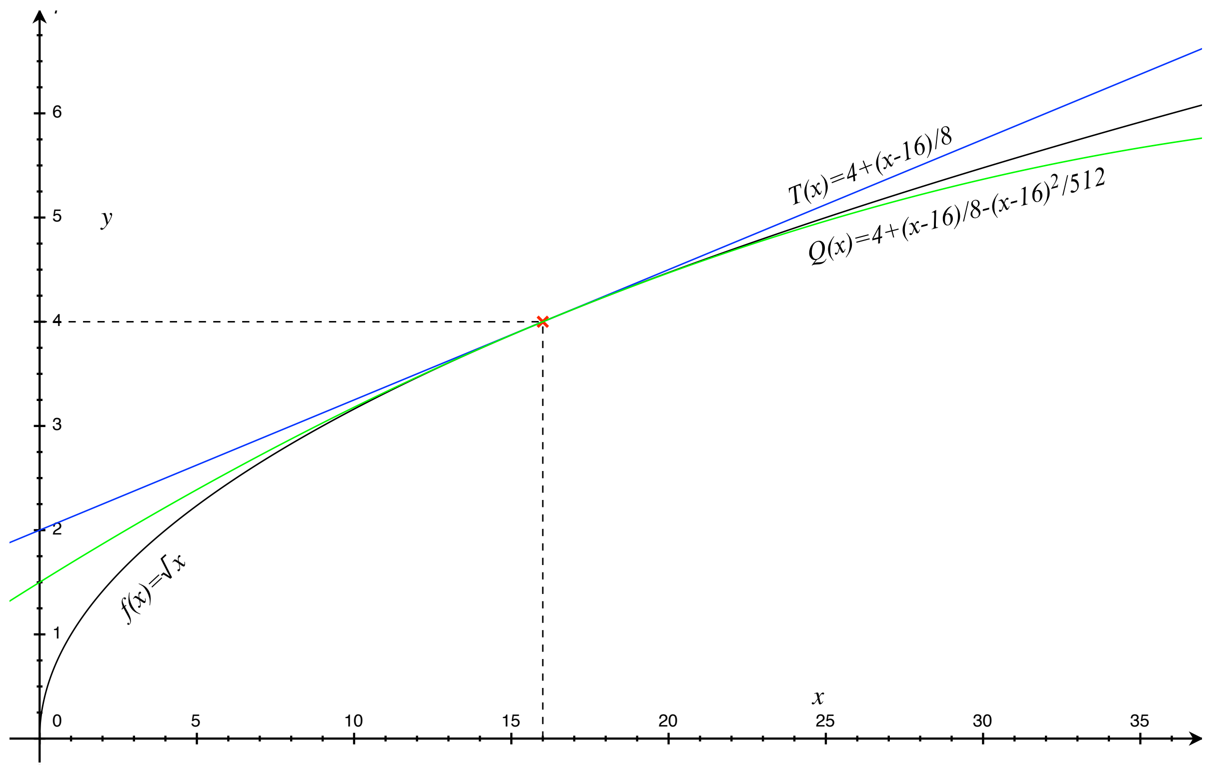
- $Q(16) = f(16) = \sqrt{16} = 4$,
- $Q'(16) = f'(16) = \dfrac{1}{2}(16)^{-1/2} = \dfrac{1}{8}$ and
- $Q''(16) = f''(16) = -\dfrac{1}{4}(16)^{-3/2} = -\dfrac{1}{256}$

So that the slope of $Q(x)$ will be changing at the same rate as the slope of $f(x) = \sqrt{x}$ at $x = 16$.

If we write $Q(x) = A + B(x-16) + C(x-16)^2$, we find that

- $Q(16) = A$, so $A = 4$.
- $Q'(x) = B + 2C(x-16)$, so $Q'(16) = B$ and $B = \frac{1}{8}$.
- $Q''(x) = 2C$, so $2C = -\frac{1}{256}$, i.e., $C = -\frac{1}{512}$.

$$\Longrightarrow \quad Q(x) = 4 + \frac{1}{8}(x-16) - \frac{1}{512}(x-16)^2.$$



Numerical comparisons:

x	$\sqrt{x}$(calculator value)	$T(x)$	$Q(x)$	$\lvert\sqrt{x} - Q(x)\rvert$
16	4	4	4	0
17	4.123105626	4.125	4.123046875	< 0.00006
15	3.872983346	3.875	3.873046875	< 0.000064
20	4.472135955	4.5	4.46875	< 0.0034
12	3.464101615	3.5	3.46875	< 0.0047
16.5	4.062019202	4.0625	4.062011719	< 0.0000075

Generalizing.

Given a differentiable function $f(x)$ and a point $x_0 \ldots$ If $x \approx x_0$, then

$$f(x) \approx T_1(x) = f(x_0) + f'(x_0)(x - x_0).$$

This is linear approximation, and the the function $T_1(x)$ is the *linear Taylor polynomial for $f(x)$ centered at x_0.*

$T_1(x)$ has the properties; (i) $T_1(x_0) = f(x_0)$ and (ii) $T_1'(x_0) = f'(x_0)$.

Intuition: If we can find a quadratic function $T_2(x)$ satisfying

$$T_2(x_0) = f(x_0), \quad T_2'(x_0) = f'(x_0) \text{ and } T_2''(x_0) = f''(x_0),$$

then $T_2(x)$ will provide a better approximation to $f(x)$ than $T_1(x)$ does.

Definition: The ***quadratic Taylor polynomial*** for the function $y = f(x)$, centered at x_0, is the function

$$T_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2.$$

This function has the properties

- $T_2(x_0) = f(x_0)$
- $T_2'(x) = f'(x_0) + f''(x_0)(x - x_0)$, so $T_2'(x_0) = f'(x_0)$
- $T_2''(x) = f''(x_0)$, so $T_2''(x_0) = f''(x_0)$

Quadratic approximation:

> *Generally speaking, if $x \approx x_0$, then $T_2(x)$ gives a better approximation to $f(x)$ than $T_1(x)$.*

Example. Find the quadratic Taylor polynomial for $f(x) = \sqrt{x}$, centered at $x_0 = 25$.

We need to find $f(25)$, $f'(25)$ and $f''(25)$...

$$f(x) = \sqrt{x} = x^{1/2} \implies f'(x) = \frac{1}{2}x^{-1/2} \text{ and } f''(x) = -\frac{1}{4}x^{-3/2},$$

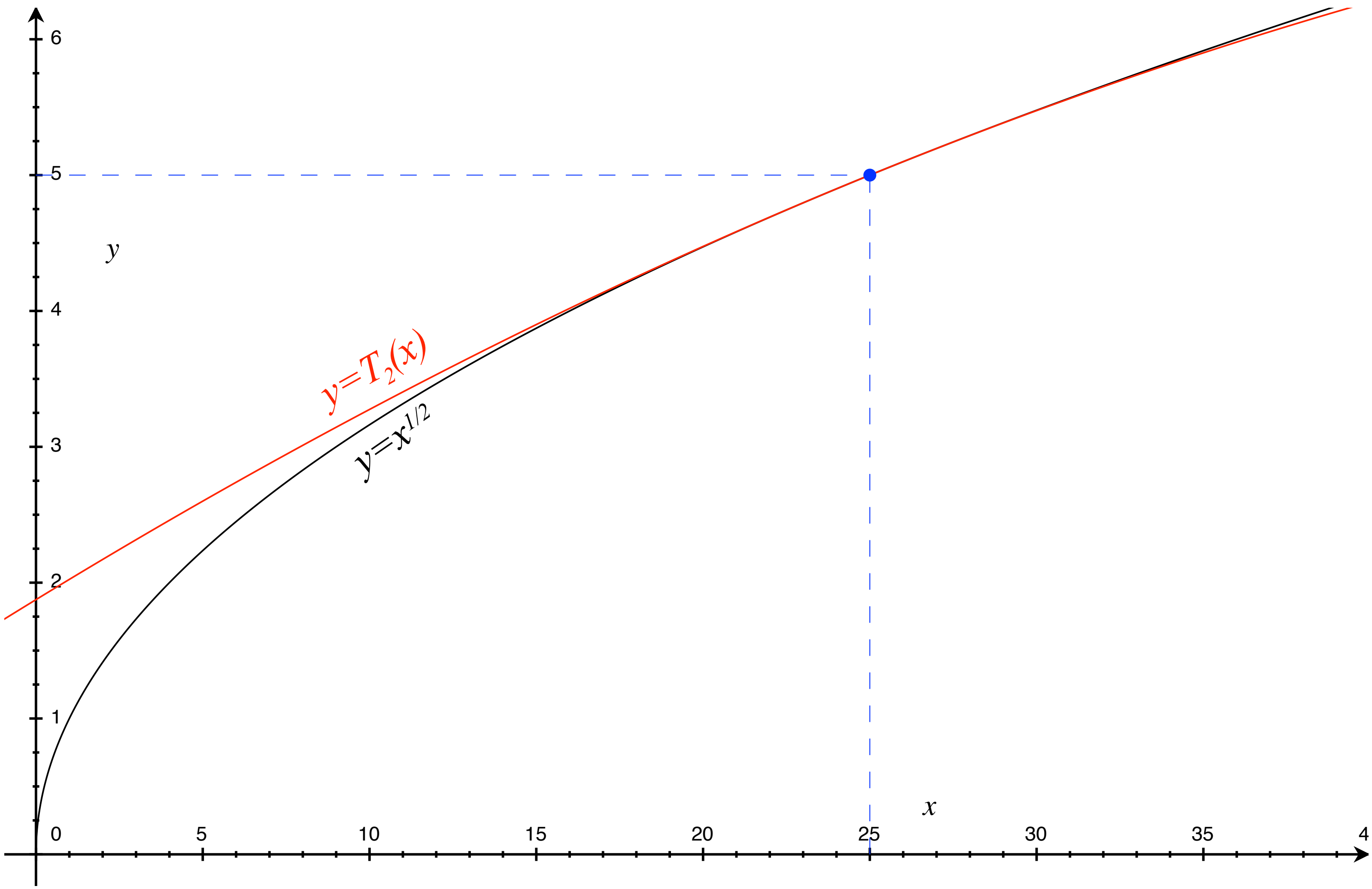
so $f(25) = 25^{1/2} = 5$ and

$$f'(25) = \frac{1}{2}25^{-1/2} = \frac{1}{10} \text{ and } f''(25) = -\frac{1}{4}25^{-3/2} = -\frac{1}{500}$$

Therefore, the quadratic Taylor polynomial for $f(x) = \sqrt{x}$, centered at $x_0 = 25$ is

$$\begin{aligned} T_2(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \\ &= 5 + \underbrace{\frac{1}{10}}_{f'(25)}(x - 25) \underbrace{-\frac{1}{1000}}_{f''(25)/2}(x - 25)^2 \end{aligned}$$

Figure 1: Quadratic approximation to $y = x^{1/2}$, centered at $x = 25$



Answer 1: *It looks great for* $20 < x < 30$*, based on the pretty picture.*

Answer 2: *Some numerical comparisons:*

x	$T_2(x)$	$\sqrt{x}$ (calculator)	$\lvert\sqrt{x} - T_2(x)\rvert$
25	5	5	0
24	4.899	4.898979...	< 0.000021
26	5.099	5.099019 ...	< 0.00002
23	4.796	4.795831 ...	< 0.00017
27	5.196	5.196152 ...	< 0.00016
20	4.475	4.472135 ...	< 0.0029
30	5.475	5.477225 ...	< 0.0023

To improve on quadratic approximation... Keep going.

The Taylor polynomial of degree n.

Given an n-times differentiable function $f(x)$ and a point x_0, the degree n Taylor polynomial for $f(x)$ centered at $x = x_0$ is given by

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Comments:

1. $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1.$

2. $T_{n+1}(x) = T_n(x) + \dfrac{f^{(n+1)}(x_0)}{(n+1)!}(x - x_0)^{n+1}$

3. $T_n(x)$ has the property that $T_n(x_0) = f(x_0)$, and

$$T_n'(x_0) = f'(x_0),\ T_n''(x_0) = f''(x_0),\ \ldots,\ T_n^{(n)}(x_0) = f^{(n)}(x_0).$$

4. The approximation $f(x) \approx T_n(x)$ is usually very accurate when x is close to x_0, especially when $|x - x_0| < 1$.

Example. Find the cubic Taylor approximation, $T_3(x)$, for $f(x) = \sqrt{x}$, centered at $x_0 = 25$.

We already know that $T_2(x) = 5 + 0.1(x - 25) - 0.001(x - 25)^2$ and

$$\frac{d^3}{dx^3}\sqrt{x} = \frac{3}{8}x^{-5/2}$$

therefore

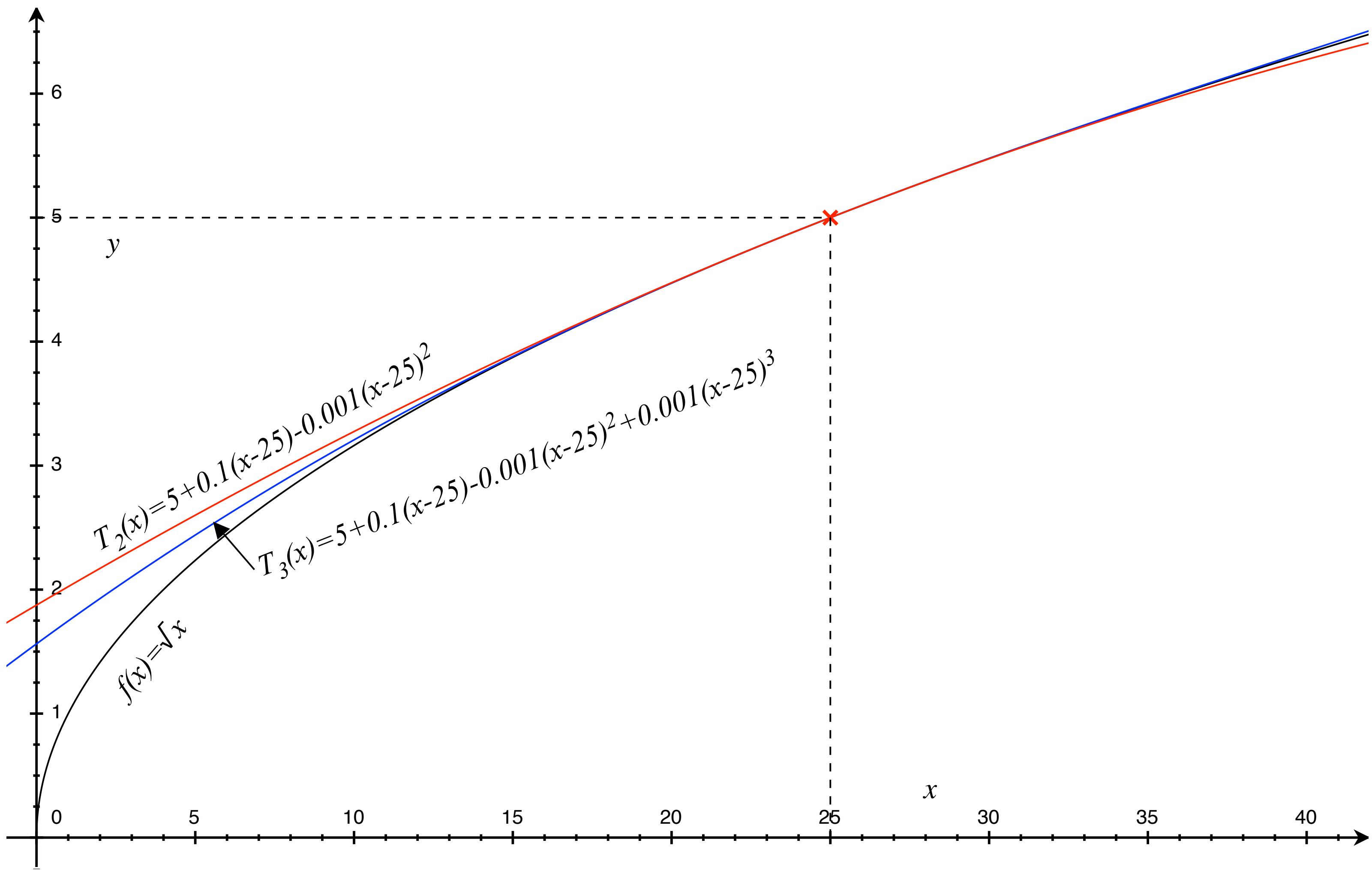
$$\left.\frac{d^3}{dx^3}\sqrt{x}\right|_{x=25} = \left.\frac{3}{8}x^{-5/2}\right|_{x=25} = \frac{3}{25000}$$

Furthermore $3! = 6$, and $(3/25000)/6 = 1/50000$, so

$$T_3(x) = 5 + 0.1(x - 25) - 0.001(x - 25)^2 + \underbrace{0.00002}_{f'''(25)/3!}(x - 25)^3$$

Question: How well does the approximation $\sqrt{x} \approx T_3(x)$ do?

Answer: Better than the quadratic approximation, as illustrated in the graph below. The cubic polynomial (blue curve) is closer to the graph $y = \sqrt{x}$ (black curve) than the quadratic polynomial (red curve), and stays closer for longer.



This improvement is also shown in the short table below.

x	$T_2(x)$	$T_3(x)$	$\sqrt{x}$ (calculator)	$\lvert\sqrt{x} - T_3(x)\rvert$
25	5	5	5	0
24	4.899	4.89898	4.898979...	< 0.000000515
26	5.099	5.09902	5.099019 ...	< 0.0000005
23	4.796	4.79584	4.795831 ...	< 0.0000085
27	5.196	5.19616	5.196152 ...	< 0.0000076